On a Method of Solving Integral Equation of Carleman Type on the Pair of Segments

L. A. Khvostchinskaya

Abstract The method is considered of solving integral equations of Carleman type on the pair of adjacent and disjoint segments. The problem is reduced to boundary problem of Riemann with piecewise constant matrix and four and five singular points. The solution is expressed via the solution of a differential equation of Fuchs class in which it was possible to define all the parameters.

Keywords Integral equations of Carleman type · The canonical matrix · Riemann boundary value problem · Differential equation of the Fuchs class

In 1823, N. Abel considered and solved an integral equation

$$\int_a^x \frac{\varphi(t)}{\sqrt{x-t}}dt = f(x), x > a,$$

which describes the movement of a material point by gravity in a vertical plane along a curve. Abel integral equations

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt = f(x), \quad 0 < \alpha < 1, \quad x > a,$$

arise when solving inverse problems in solid state physics (determining the potential energy from the oscillation period or restoring the scattering field from the effective glow in classical mechanics). Abel integral equation with constant limits

$$\int_a^b \frac{\varphi(t)}{|x-t|^{1-\alpha}}dt = f(x), \quad 0 < \alpha < 1, \quad a < x < b, \tag{1}$$

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it was decided by Carleman [1]. The unique solution to Eq. (1) is given by the formula [2]

$$\varphi(x) = \frac{tg\frac{\pi\alpha}{2}}{2\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^\alpha} - \frac{\sin^2\frac{\pi\alpha}{2}}{\pi^2}\frac{d}{dx}\int_a^x \left(\frac{b-t}{t-a}\right)^{\frac{\alpha}{2}} \frac{dt}{(x-t)^\alpha}\cdot$$
$$\cdot\frac{d}{dt}\int_a^t \frac{d\tau}{(t-\tau)^{1-\alpha}}\int_\tau^b \left(\frac{s-a}{b-s}\right)^{\frac{\alpha}{2}} \frac{f(s)\,ds}{(s-\tau)^\alpha}. \quad (2)$$

Consider the integral equation of Carleman type

$$\int_{L_1} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} + \int_{L_2} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = f(x), \quad (3)$$

where α_1, α_2 are given real numbers, $0 < \alpha_k < 1$, $k = 1, 2$, $\alpha_1 \neq \alpha_2$, in the following two cases:

1. on a pair of adjacent segments $L_1 = [a_1, a_2]$, $L_2 = [a_2, a_3]$,
2. on a pair of disjoint segments $L_1 = [a_1, b_1]$, $L_2 = [a_2, b_2]$, $b_1 \neq a_2$.

The solution $\varphi(z)$ of problems (2) will be sought in the class of functions satisfying the Hölder condition inside the segments and that are integrable at the ends of segments, $f(x) = f_k(x)$, $x \in L_k$, $k = 1, 2$, are corresponding Holder functions.

Solution of Eq. (3) in the case $a_3 = \infty$ was constructed in [3] explicitly and expressed in terms of hypergeometric functions. It was also noted here that when $a_3 \neq \infty$ solution of Eq. (3) is much more complicated.

Equation (3) is a generalization of the Carleman equation (2). Let us reduce the integral equation (3) to the Riemann vector-matrix boundary value problem and construct a solution of this equation in each of the two cases, using the results of [4–7].

We construct the solution of the integral equation (3) on a pair of adjacent segments:

$$\int_{a_1}^{a_2} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = f(x), \quad a_1 < x < a_3. \quad (4)$$

We write Eq. (4) in the form of a system of three equations

$$\int_{a_1}^{a_2} \frac{\varphi_1(t)\,dt}{|x-t|^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi_2(t)\,dt}{(t-x)^{\alpha_2}} = f_1(x), \quad a_1 < x < a_2,$$

$$\int_{a_1}^{a_2} \frac{\varphi_1(t)\,dt}{(x-t)^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi_2(t)\,dt}{|x-t|^{\alpha_2}} = f_2(x), \quad a_2 < x < a_3, \quad (5)$$

$$\int_{a_1}^{a_2} \frac{\varphi_1(t)\,dt}{(x-t)^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi_2(t)\,dt}{(x-t)^{\alpha_2}} = 0, \quad a_3 < x < \infty.$$

We introduce two new unknown functions

$$\Phi_k(z) = \int_{a_k}^{a_{k+1}} \frac{\varphi_k(t)\,dt}{(t-z)^{\alpha_k}}, \quad k = 1, 2,$$

which are analytic in the complex plane z with the cut along the ray (a_1, ∞). Find the limiting values of these functions on the banks of the section.

For $a_1 < x < a_2$ we get $\Phi_1^{\pm}(x) = e^{\pm\pi i\alpha_1} \int_{a_1}^{x} \frac{\varphi(t)dt}{(x-t)^{\alpha_1}} + \int_{x}^{a_2} \frac{\varphi(t)dt}{(x-t)^{\alpha_1}}$, where we find

$$\int_{a_1}^{a_2} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} = \frac{e^{\pi i\alpha_1}\Phi_1^+(x) + \Phi_1^-(x)}{1+e^{\pi i\alpha_1}}, \; \Phi_2^+(x) = \Phi_2^-(x) = \int_{a_2}^{a_3} \frac{\varphi(t)\,dt}{(t-x)^{\alpha_2}} \tag{6}$$

Similarly for $a_2 < x < a_3$ we get

$$\int_{a_2}^{a_3} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = \frac{e^{\pi i\alpha_2}\Phi_2^+(x) + \Phi_2^-(x)}{1+e^{\pi i\alpha_2}}, \; \Phi_1^+(x) = e^{-2\pi i\alpha_1}\Phi_1^-(x) = = e^{-\pi i\alpha_1} \int_{a_1}^{a_2} \frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}}. \tag{7}$$

For $a_3 < x < \infty$

$$\Phi_1^+(x) = e^{-\pi i\alpha_1}\Phi_1^-(x), \quad \Phi_2^+(x) = e^{-\pi i\alpha_2}\Phi_2^-(x). \tag{8}$$

Using formulas (6)–(8), we rewrite system (5) as boundary conditions for two functions $\Phi_1(z)$ and $\Phi_2(z)$ [8–10]:

$$\begin{cases} \Phi_1^+(x) = -e^{\pi i\alpha_1}\Phi_1^-(x) - \left(1+e^{-\pi i\alpha_1}\right)\Phi_2^-(x) + \left(1+e^{-\pi i\alpha_1}\right) f_1(x), \\ \Phi_2^+(x) = \Phi_2^-(x), \quad a_1 < x < a_2, \end{cases}$$

$$\begin{cases} \Phi_1^+(x) = e^{-2\pi i\alpha_1}\Phi_1^-(x), \quad a_2 < x < a_3, \\ \Phi_2^+(x) = -e^{-\pi i\alpha_1}\left(1+e^{-\pi i\alpha_2}\right)\Phi_1^-(x) - e^{-\pi i\alpha_2}\Phi_2^-(x) + \left(1+e^{-\pi i\alpha_2}\right) f_2(x), \end{cases}$$

$$\begin{cases} \Phi_1^+(x) = e^{-2\pi i\alpha_1}\Phi_1^-(x), \\ \Phi_2^+(x) = e^{-2\pi i\alpha_2}\Phi_2^-(x), \end{cases} \quad a_3 < x < \infty.$$

So we have obtained the Riemann boundary value problem for the vector function $\Phi(z) = \left(\Phi_1(z), \Phi_2(z)\right)$ with a piecewise constant matrix and four singular points a_1, a_2, a_3, ∞:

$$\Phi^+(x) = A_k\Phi^-(x) + F_k(x), \quad a_k < x < a_{k+1}, k = 1, 2, 3; a_4 = \infty, \tag{9}$$

$$A_1 = \begin{pmatrix} -e^{-\pi i\alpha_1} & -\left(1+e^{-\pi i\alpha_1}\right) \\ 0 & 1 \end{pmatrix}, \quad F_1(x) = \begin{pmatrix} \left(1+e^{-\pi i\alpha_1}\right) f_1(x) \\ 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} e^{-\pi i \alpha_1} & 0 \\ -e^{-\pi i \alpha_1}\left(1+e^{-\pi i \alpha_2}\right) & -e^{-\pi i \alpha_2} \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 \\ \left(1+e^{-\pi i \alpha_2}\right) f_2(x) \end{pmatrix},$$

$$A_3 = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 \\ 0 & e^{-2\pi i \alpha_2} \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solution of problems (9) will be sought in the class of functions that are bounded as $z \to a_k$, $k = 1, 2, 3$, and disappearing at infinity.

In order to solve the inhomogeneous boundary value problem (9), it is necessary to construct a canonical matrix $X(z)$ corresponding homogeneous boundary value problem. The columns of the matrix $X(z)$ consist of linearly independent solutions of a homogeneous boundary problem

$$\Phi^+(x) = A_k \Phi^-(x), \quad a_k < x < a_{k+1}, \quad k = 1, 2, 3; \quad a_4 = \infty, \tag{10}$$

and orders p_1, p_2 first and second columns $X(z)$ at infinity satisfy inequality $p_1 \leq p_2$. The matrix $X(z)$ has the following properties [11]:

1. $\det X(z) \neq 0$ for $\forall z \neq a_k$ $(k = 1, 2, 3)$;
2. the columns of the matrix $X(z)$ belong to the selected class of functions;
3. the order of the determinant $X(z)$ is equal to the sum of the orders of its columns.

If the matrix $X(z)$ multiply on the left by a constant nondegenerate second-order upper triangular matrix T, then the matrix $X(z)\,T$ will also be canonical, since the orders of the determinant and the columns of the matrix will not change.

The canonical matrix $\boldsymbol{X}(x)$ of homogeneous boundary value problem (10) is a solution of a system of differential equations of Fuchs class with four singular points a_1, a_2, a_3, ∞ [12]:

$$\frac{dX}{dz} = X \sum_{k=1}^{3} \frac{U_k}{z - a_k}, \tag{11}$$

moreover, differential matrices U_k like matrices $W_k = \frac{1}{2\pi i} \ln A_{k-1} A_k^{-1}, k = 1, \ldots, 4, A_0 = A_4 = E$. Matrices $V_k = A_{k-1} A_k^{-1}$, $k = 1, \ldots, 4$, form a monodromy group of a differential equation (11) [13–15].

Find differential matrices U_k systems (11) by the "logarithmization method of matrix product" of the 2nd order [4].

Let V_1, V_2 be constant non-degenerate matrices of the 2nd order, $V_3 = V_1 V_2$. Equality $\ln\ (V_1 V_2) = \ln V_1 + \ln V_2$ is valid only for transitive matrices. Denote by $\alpha_k,\ \beta_k$ the characteristic numbers of matrices V_k and by $\rho_k = \frac{1}{2\pi i} \ln \alpha_k$, $\sigma_k = \frac{1}{2\pi i} \ln \beta_k$ the characteristic numbers of matrices $W_k = \frac{1}{2\pi i} \ln V_k$, $k = 1, 2,3$. Fix any branches of logarithms $\rho_1, \sigma_1, \rho_2, \sigma_2$ so that $|Re\,(\rho_k - \sigma_k)| < 1$, $k = 1, 2$. Then the branches of logarithms for ρ_3, σ_3 should be consistent and selected from the condition $\rho_1 + \sigma_1 + \rho_2 + \sigma_2 = \rho_3 + \sigma_3$.

If $\rho_3 \neq \sigma_3$, then the matrix $S = \begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ uniquely accurate to a similarity transformation using a diagonal matrix can be represented as the sum of two matrices $S = S_1 + S_2$, where $S_k \sim W_k$, $k = 1, 2$. The last equality can be written as

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \frac{\rho_1\sigma_1-(\rho_3-\rho_2)(\rho_3-\sigma_2)}{\sigma_3-\rho_3} & \frac{(\rho_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3}c \\ \frac{\rho_2\sigma_2-(\rho_3-\sigma_1)(\sigma_3-\rho_1)}{c(\sigma_3-\rho_3)} & \frac{(\sigma_3-\rho_2)(\sigma_3-\sigma_2)-\rho_1\sigma_1}{\sigma_3-\rho_3} \end{pmatrix} +$$
$$+ \begin{pmatrix} \frac{\rho_2\sigma_2+(\rho_3-\rho_1)(\rho_3-\sigma_1)}{\sigma_3-\rho_3} & \frac{\rho_2\sigma_2-(\rho_3-\rho_1)(\sigma_3-\sigma_1)}{\sigma_3-\rho_3}c \\ \frac{(\rho_3-\sigma_1)(\sigma_3-\rho_1)-\rho_2\sigma_2}{c(\sigma_3-\rho_3)} & \frac{(\sigma_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3} \end{pmatrix} \quad (12)$$

where c is an arbitrary constant. If $\rho_3 = \rho_1 + \rho_2, \sigma_3 = \sigma_1 + \sigma_2$, then matrices V_1, V_2 are reduced by a single similarity transformation to a triangular form and simpler matrix representations take place S:

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \rho_1 & c \\ 0 & \sigma_1 \end{pmatrix} + \begin{pmatrix} \rho_2 & -c \\ 0 & \sigma_2 \end{pmatrix}, \quad (13)$$

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \rho_1 & 0 \\ c & \sigma_1 \end{pmatrix} + \begin{pmatrix} \rho_2 & 0 \\ -c & \sigma_2 \end{pmatrix}. \quad (14)$$

Let V_1, V_2, V_3 be constant non-degenerate matrices of the 2nd order, $V_4 = V_1V_2V_3$. Denote by α_k, β_k the characteristic numbers of matrices V_k and by $\rho_k = \frac{1}{2\pi i}\ln\alpha_k$, $\sigma_k = \frac{1}{2\pi i}\ln\beta_k$ the characteristic numbers of matrices $W_k = \frac{1}{2\pi i}\ln U_k$, $k = 1, \ldots, 4$, where the branches of logarithms satisfy the conditions $|Re\,(\rho_k - \sigma_k)| < 1$ and

$$\sum_{k=1}^{3}(\rho_k + \sigma_k) = \rho_4 + \sigma_4. \quad (15)$$

If $\rho_4 \neq \sigma_4$, then the matrix W_4 is reduced to diagonal Jordan form $S = \begin{pmatrix} \rho_4 & 0 \\ 0 & \sigma_4 \end{pmatrix}$. Representation of the matrix S as the sum of three matrices $S = S_1 + S_2 + S_3$, where $S_k \sim W_k$, $k = 1, 2, 3$, we get from the formulas (12)–(14). We write the product of matrices $V_1 \cdot V_2 \cdot V_3$ in the form of multiplication of two matrices as follows:

$$V_4 = V_1 \cdot V_2 \cdot V_3 = V_1 \cdot (V_2 \cdot V_3) = V_1 \cdot V_{23},$$

$$V_4 = V_1 \cdot V_2 \cdot V_3 = (V_1 \cdot V_2) \cdot V_3 = V_{12} \cdot V_3.$$

Therefore, we need to find the characteristic numbers α_{12}, β_{12} and α_{23}, β_{23} respectively matrices V_{12}, V_{23} and numbers $\rho_{12} = \frac{1}{2\pi i}\ln\alpha_{12}$, $\sigma_{12} = \frac{1}{2\pi i}\ln\beta_{12}$,

$\rho_{23} = \frac{1}{2\pi i} \ln \alpha_{23}$, $\sigma_{23} = \frac{1}{2\pi i} \ln \beta_{23}$, whose branches are chosen from the conditions $\rho_{12}+\sigma_{12} = \rho_1+\sigma_1+\rho_2+\sigma_2$, $|Re\,(\rho_{12}-\sigma_{12})| < 1$, $\rho_{23}+\sigma_{23} = \rho_2+\sigma_2+\rho_3+\sigma_3$, $|Re\,(\rho_{23}-\sigma_{23})| < 1$.

We write the matrices V_1, V_2, V_3, V_4 monodromy groups of problem (10) and their characteristic numbers λ_k, μ_k. $\left(k = \overline{1,4}\right)$:

$$V_1 = A_1^{-1} = \begin{pmatrix} -e^{\pi i\alpha_1} & -\left(1+e^{\pi i\alpha_1}\right) \\ 0 & 1 \end{pmatrix},$$

$$V_2 = A_1A_2^{-1} = \begin{pmatrix} -e^{\pi i\alpha_1} + \left(1+e^{\pi i\alpha_1}\right)\left(1+e^{\pi i\alpha_2}\right) & e^{\pi i\alpha_2}\left(1+e^{-\pi i\alpha_1}\right) \\ -e^{\pi i\alpha_1}\left(1+e^{\pi i\alpha_2}\right) & -e^{\pi i\alpha_2} \end{pmatrix},$$

$$V_3 = A_2A_3^{-1} = \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha_1}\left(1+e^{-\pi i\alpha_2}\right) & -e^{\pi i\alpha_2} \end{pmatrix},$$

$$V_4 = A_3, \lambda_1 = -e^{\pi i\alpha_1}, \mu_1 = 1;$$

$$\lambda_2 = -1, \mu_2 = -e^{\pi i(\alpha_1+\alpha_2)}; \lambda_3 = 1, \mu_3 = -e^{-\pi i\alpha_2}; \lambda_4 = e^{-2\pi i\alpha_1},$$

$$\mu_4 = e^{-2\pi i\alpha_2}.$$

Next we find the numbers $\rho_k = \frac{1}{2\pi i}\ln\lambda_k$, $0 \le Re\lambda_k < 1$, $\sigma_k = \frac{1}{2\pi i}\ln\mu_k$, $0 \le Re\sigma_k < 1$, $k = \overline{1,4}$, $0 \le Re\lambda_k < 1$, $\rho_1 = \frac{1+\alpha_1}{2}$, $\sigma_1 = 0$; $\rho_2 = \frac{1}{2}$, $\sigma_2 = \frac{\alpha_1+\alpha_2+1}{2}$; $\rho_3 = 0$, $\sigma_3 = \frac{1+\alpha_2}{2}$; $\rho_4 = 1-\alpha_1$, $\sigma_4 = 1-\alpha_2$, $\Delta = \sum_{k=1}^{4}(\rho_k+\sigma_k) = 4$.

The behavior of the solution of problem (9) at infinity determine the numbers $\rho = \rho_4 - 1 = -\alpha_1$, $\sigma = \sigma_4 - 2 = -\alpha_2 - 1$, if $\alpha_1 > \alpha_2$ and $\rho = \rho_4 - 2 = -\alpha_1 - 1$, $\sigma = \sigma_4 - 1 = -\alpha_2$, if $\alpha_1 < \alpha_2$.

Numbers ρ_k, σ_k $(k = 1, 2, 3)$, ρ, σ satisfy the Fuchs relation:

$$\sum_{k=1}^{3}(\rho_k+\sigma_k) + \rho + \sigma = 1. \tag{16}$$

The total index κ and partial indices æ$_1$, æ$_2$ of the problem (9) are respectively equal æ $= -\Delta = -4$, æ$_1$ = æ$_2$ $= -2$, those problem (9) will be solvable if four solvability conditions are satisfied.

We also find the characteristic numbers λ_{12}, μ_{12} and λ_{23}, μ_{23} of the matrices

$$V_{12} = V_1 \cdot V_2 = A_2^{-1} = \begin{pmatrix} e^{2\pi i\alpha_1} & 0 \\ -e^{\pi i(\alpha_1+\alpha_2)}\left(1-e^{-\pi i\alpha_2}\right) & -e^{\pi i\alpha_2} \end{pmatrix},$$

$$\lambda_{12} = -e^{\pi i\alpha_1}, \mu_{12} = -e^{\pi i\alpha_2};$$

$$V_{23} = V_2 \cdot V_3 = A_1 \cdot A_3^{-1} = \begin{pmatrix} -e^{\pi i \alpha_1} & -e^{2\pi i \alpha_2}\left(1 + e^{-\pi i \alpha_1}\right) \\ 0 & e^{2\pi i \alpha_2} \end{pmatrix},$$

$$\lambda_{23} = -e^{\pi i \alpha_1},\, \mu_{23} = e^{2\pi i \alpha_2}.$$

Branches of logarithms of numbers $\rho_{k,k+1} = \frac{1}{2\pi i} \ln \lambda_{k,k+1}$ and $\sigma_{k,k+1} = \frac{1}{2\pi i} \ln \mu_{k,k+1}$ should be conditions

$$\rho_{12}+\sigma_{12} = \rho_1+\sigma_1+\rho_2+\sigma_2 = 1+\alpha_1+\frac{1+\alpha_2}{2} \;\Rightarrow\; \rho_{12} = 1+\alpha_1,\; \sigma_{12} = \frac{1+\alpha_2}{2},$$

$$\rho_{23}+\sigma_{23} = \rho_2+\sigma_2+\rho_3+\sigma_3 = 1+\alpha_2+\frac{1+\alpha_1}{2} \;\Rightarrow\; \rho_{23} = \frac{1+\alpha_1}{2},\; \sigma_{12} = 1+\alpha_2.$$

Comparing formulas (15) and (16), we notice that $\rho_4 + \sigma_4 = 1 - \rho - \sigma$, those

$\rho_4 = 1 - \rho$, $\sigma_4 = -\sigma$, if $\alpha_1 > \alpha_2$,
$\rho_4 = 1 - \sigma$, $\sigma_4 = -\rho$, if $\alpha_1 < \alpha_2$.

Denote by $S = \begin{pmatrix} -\min(\rho,\sigma) & 0 \\ 0 & 1-\max(\rho,\sigma) \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix}$.

Imagine the matrix S as the sum of three matrices using the representations (13) and (14):

$$S = S_1 + S_2 + S_3 = S_1 + S_{23} = S_{12} + S_3, \tag{17}$$

where

$$S_k \sim \frac{1}{2\pi i} \ln V_k,\, S_{12} \sim \frac{1}{2\pi i} \ln V_{12},\, S_{23} \sim \frac{1}{2\pi i} \ln V_{23}. S_1 + S_{23} = S \;\Rightarrow$$

$$\begin{pmatrix} \frac{1+\alpha_1}{2} & c \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1+\alpha_1}{2} & -c \\ 0 & \alpha_2+1 \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix},\, S_{12} + S_3 = S \;\Rightarrow$$

$$\begin{pmatrix} \alpha_1+1 & 0 \\ d & \frac{\alpha_2+1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -d & \frac{1+\alpha_2}{2} \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix},$$

where c, d are arbitrary constants. From (17) it follows that

$$S_2 = S_{23} - S_3 = S_{12} - S_1 = \begin{pmatrix} \frac{1+\alpha_1}{2} & -c \\ d & \frac{1+\alpha_2}{2} \end{pmatrix}.$$

Since $S_2 \sim \frac{1}{2\pi i} \ln V_2$, that $\det S_2 = \rho_2 \cdot \sigma_2$, or $\frac{1+\alpha_1}{2} \cdot \frac{1+\alpha_2}{2} + c \cdot d = \frac{1}{2} \cdot \frac{1+\alpha_1+\alpha_2}{2} \Rightarrow c \cdot d = -\frac{1}{4}\alpha_1 \cdot \alpha_2$. Matrices S_k $(k = 1, 2, 3)$ are differential matrices of system (11), which takes the form

$$\frac{dX}{dz} = X \left[\frac{\begin{pmatrix} (1+\alpha_1)/2 & c \\ 0 & 0 \end{pmatrix}}{z - a_1} + \frac{\begin{pmatrix} (1+\alpha_1)/2 & -c \\ -\alpha_1 \cdot \alpha_2 / 4c & (1+\alpha_2)/2 \end{pmatrix}}{z - a_2} + \right.$$
$$\left. + \frac{\begin{pmatrix} 0 & 0 \\ \alpha_1 \cdot \alpha_2 / 4c & (1+\alpha_2)/2 \end{pmatrix}}{z - a_3} \right], \tag{18}$$

where c is an arbitrary constant.

Let be $X(z) = \begin{pmatrix} u(z) & u_1(z) \\ v(z) & v_1(z) \end{pmatrix}$. Substituting this matrix into Eq. (18), we obtain the following system of differential equations connecting the functions $u(z)$ and $u_1(z)$:

$$\begin{aligned} u' &= \rho_1 \left(\tfrac{1}{z-a_1} + \tfrac{1}{z-a_2} \right) u + d \left(\tfrac{1}{z-a_3} - \tfrac{1}{z-a_2} \right) u_1, \\ u_1' &= c \left(\tfrac{1}{z-a_1} - \tfrac{1}{z-a_2} \right) u + \sigma_2 \left(\tfrac{1}{z-a_3} + \tfrac{1}{z-a_3} \right) u_1 \end{aligned} \tag{19}$$

Functions $v(z)$ and $v_1(z)$are also solutions of the system (19). Express the function from the first equation of system (19)

$$u_1 = \frac{(z-a_2)(z-a_3)}{d(a_3-a_2)} \left[u' - \rho_1 \left(\frac{1}{z-a_1} + \frac{1}{z-a_2} \right) u \right]$$

and substitute it into the second equation. We obtain a second-order differential equation whose fundamental system of solutions are functions $u(z)$and$v(z)$. This is a differential equation of Fuchs class with four singular points a_1, a_2, a_3, ∞:

$$\begin{aligned} & u'' - \tfrac{1}{2} \left(\tfrac{\alpha_1+1}{z-a_1} + \tfrac{\alpha_1+\alpha_2}{z-a_2} + \tfrac{\alpha_2-1}{z-a_3} \right) u' + \tfrac{1}{4} \left(\tfrac{2(\alpha_1+1)}{(z-a_1)^2} + \tfrac{\alpha_1+\alpha_2+1}{(z-a_2)^2} + \right. \\ & \left. + \tfrac{(4\alpha_1\alpha_2+3(\alpha_1-\alpha_2-1))z+(\alpha_1-\alpha_2-2\alpha_1\alpha_2+1)(a_1-a_3)+(\alpha_1-\alpha_2+1)a_2}{(z-a_1)(z-a_2)(z-a_3)} \right) u = 0 \end{aligned} \tag{20}$$

In the neighborhood of each singular point a_k $(k = 1, 2, 3)$ Eq. (20) has 2 linearly independent solutions, representable by series of the form

$$\begin{aligned} u_k(z) &= (z-a_k)^{\rho_k} \sum_{n=0}^{\infty} c_n^{(k)} (z-a_k)^n, \\ v_k(z) &= (z-a_k)^{\sigma_k} \sum_{n=0}^{\infty} d_n^{(k)} (z-a_k)^n, \end{aligned} \tag{21}$$

whose coefficients are found directly from the recurrence relations after substituting the series in the equation. The canonical matrix of the problem (9) in the neighborhood of each singular point is given by the formula

$$X(z) = D_k \begin{pmatrix} u_k & (z-a_2)(z-a_3)\left[u_k' - \frac{\alpha_1+1}{2}\left(\frac{1}{z-a_1}+\frac{1}{z-a_2}\right)u_k\right] \\ v_k & (z-a_2)(z-a_3)\left[v_k' - \frac{\alpha_1+1}{2}\left(\frac{1}{z-a_1}+\frac{1}{z-a_2}\right)v_k\right] \end{pmatrix},$$

$k = 1, 2, 3$, where D_k are matrices transforming the matrices V_k to a Jordan form. The solution of the boundary value problem (9) is found by the formula

$$\Phi(z) = \frac{1}{2\pi i} X(z) \sum_{k=1}^{3} \int_{a_k}^{a_{k+1}} \left[X^+(x)\right]^{-1} F_k(x) \frac{dx}{x-z} =$$

$$= \frac{1}{2\pi i} X(z) \left[\int_{a_1}^{a_2} \left[X^+(x)\right]^{-1} F_1(x) \frac{dx}{x-z} + \int_{a_2}^{a_3} \left[X^+(x)\right]^{-1} F_2(x) \frac{dx}{x-z}\right].$$

Considering that $X^+(x) = A_k X^-(x), a_k < x < a_{k+1}, k = 1, 2, 3,$ and applying the Sokhotsky formulas, as well as formulas (6) and (7), we find the integrals

$$\int_{a_k}^{a_{k+1}} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_k}} = g_k(x), \quad k = 1, 2. \tag{22}$$

Reversing equations (22) using formulas (2), we obtain a unique solution to the integral equation (4) when two matrix solvability conditions are satisfied

$$\int_{a_1}^{a_2} \left[X^+(x)\right]^{-1} F_1(x)\, x^k dx + \int_{a_2}^{a_3} \left[X^+(x)\right]^{-1} F_2(x)\, x^k dx = 0, k = 1, 2.$$

We now consider the Carleman integral equation on a pair disjoint segments.

$$\int_{a_1}^{b_1} \frac{\varphi(t)}{|x-t|^{\alpha_1}} dt + \int_{a_2}^{b_2} \frac{\varphi(t)}{|x-t|^{\alpha_2}} dt = f(x), x \in [a_1, b_1] \bigcup [a_2, b_2], \tag{23}$$

where α_1, α_2 are given real numbers, $0 < \alpha_k < 1, k = 1, 2, \alpha_1 \neq \alpha_2, a_1 < b_1 < a_2 < b_2$.

We introduce two new unknown functions

$$\Phi_k(z) = \int_{a_k}^{b_k} \frac{\varphi_k(t)}{(t-z)^{\alpha_k}} dt, \quad k = 1, 2, \quad \varphi_k(t) = \varphi(t), \quad t \in [a_k, b_k],$$

which are analytic in the complex plane z with a cut along the ray (a_1, ∞). Find the limiting values of these functions on the banks of the section.

For $a_1 < x < b_1$ we get

$$\Phi_1^{\pm}(x) = e^{\pm\pi i\alpha_1}\int_{a_1}^{x}\frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}} + \int_{x}^{b_1}\frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}},$$

where we find

$$\int_{a_1}^{b_1}\frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} = \frac{e^{\pi i\alpha_1}\Phi_1^{+}(x) + \Phi_1^{-}(x)}{1+e^{\pi i\alpha_1}},$$

$$\Phi_2^{+}(x) = \Phi_2^{-}(x) = \int_{a_2}^{b_2}\frac{\varphi(t)\,dt}{(t-x)^{\alpha_2}}. \tag{24}$$

Similarly for $a_2 < x < b_2$ we get

$$\int_{a_2}^{b_2}\frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = \frac{e^{\pi i\alpha_2}\Phi_2^{+}(x) + \Phi_2^{-}(x)}{1+e^{\pi i\alpha_2}},\ \Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x) =$$

$$= e^{-\pi i\alpha_1}\int_{a_1}^{b_1}\frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}} \tag{25}$$

For $b_1 < x < a_2$

$$\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\ \Phi_2^{+}(x) = \Phi_2^{-}(x).$$

For $b_2 < x < \infty$

$$\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\ \Phi_2^{+}(x) = e^{-2\pi i\alpha_2}\Phi_2^{-}(x).$$

We write the system of boundary conditions for two functions $\Phi_1(z)$ and $\Phi_2(z)$:

$$\begin{cases}\Phi_1^{+}(x) = -e^{\pi i\alpha_1}\Phi_1^{-}(x) - \left(1+e^{-\pi i\alpha_1}\right)\Phi_2^{-}(x) + \left(1+e^{-\pi i\alpha_1}\right)f_1(x),\\ \Phi_2^{+}(x) = \Phi_2^{-}(x), \quad a_1 < x < b_1,\end{cases}$$

$$\begin{cases}\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\\ \Phi_2^{+}(x) = \Phi_2^{-}(x),\end{cases} \quad b_1 < x < a_2.$$

$$\begin{cases}\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x), \quad a_2 < x < b_2,\\ \Phi_2^{+}(x) = -e^{-\pi i\alpha_1}\left(1+e^{-\pi i\alpha_2}\right)\Phi_1^{-}(x) - e^{-\pi i\alpha_2}\Phi_2^{-}(x) + \left(1+e^{-\pi i\alpha_2}\right)f_2(x),\end{cases}$$

$$\begin{cases}\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\\ \Phi_2^{+}(x) = e^{-2\pi i\alpha_2}\Phi_2^{-}(x),\end{cases} \quad b_2 < x < \infty.$$

So we have obtained the Riemann boundary value problem for the vector function $\Phi(z) = (\Phi_1(z), \Phi_2(z))$ with a piecewise constant matrix and five singular points $a_1, b_1, a_2, b_2, \infty$:

$$\Phi^+(x) = A_k \Phi^-(x) + F_k(x), \quad x \in l_k, \quad k = 1,2,3,4; \tag{26}$$

$$l_1 \in (a_1, b_1), l_2 \in (b_1, a_2), l_3 \in (a_2, b_2), l_4 \in (b_2, \infty),$$

$$A_1 = \begin{pmatrix} -e^{-\pi i \alpha_1} & -\left(1 + e^{-\pi i \alpha_1}\right) \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 \\ 0 & 1 \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} \left(1 + e^{-\pi i \alpha_1}\right) f_1(x) \\ 0 \end{pmatrix}, A_3 = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 \\ -e^{-\pi i \alpha_1}\left(1 + e^{-\pi i \alpha_2}\right) & -e^{-\pi i \alpha_2} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 \\ 0 & e^{-2\pi i \alpha_2} \end{pmatrix}, F_3(x) = \begin{pmatrix} 0 \\ \left(1 + e^{-\pi i \alpha_2}\right) f_2(x) \end{pmatrix},$$

$$F_2(x) = F_4(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_k(x) = f(x), \quad x \in [a_k, b_k], \quad k = 1, 2.$$

Find the characteristic numbers $\lambda_k, \quad \mu_k, \quad k = \overline{1,5}$, of the monodromy matrices $V_k = A_{k-1} A_k^{-1}$, $A_0 = A_5 = E$ and numbers $\rho_k = \frac{1}{2\pi i} \ln \lambda_k$, $\sigma_k = \frac{1}{2\pi i} \ln \mu_k$, $0 \le Re \rho_k < 1$, $0 \le Re \sigma_k < 1$, and characteristic numbers and corresponding logarithms of matrices $V_1 V_2$, $V_3 V_4$, $V_1 V_2 V_3$, $V_2 V_3 V_4$:

$$V_1 = A_1^{-1}, V_2 = A_1 A_2^{-1}, \lambda_1 = \lambda_2 = -e^{\pi i \alpha_1}, \mu_1 = \mu_2 = 1;$$

$$\rho_1 = \rho_2 = (1 + \alpha_1)/2, \sigma_1 = \sigma_2 = 0,$$

$$V_3 = A_2 A_3^{-1}, V_4 = A_3 A_4^{-1}, \lambda_3 = \lambda_4 = 1, \mu_3 = \mu_4 = -e^{-\pi i \alpha_2},$$

$$\rho_3 = \rho_4 = 0, \rho_3 = \rho_4 = (1 + \alpha_2)/2,$$

$$V_5 = A_4, \lambda_5 = e^{-2\pi i \alpha_1}, \mu_5 = e^{-2\pi i \alpha_2}, \rho_5 = 1 - \alpha_1, \sigma_5 = 1 - \alpha_2,$$

$$V_{12} = V_1 \cdot V_2 = A_2^{-1}, \lambda_{12} = e^{2\pi i \alpha_1}, \mu_{12} = 1, \rho_{12} = \alpha_1, \ \sigma_{12} = 1,$$

$$V_{34} = V_3 \cdot V_4 = A_2 \cdot A_4^{-1}, \lambda_{34} = 1, \mu_{34} = e^{2\pi i \alpha_2}, \ \rho_{34} = 1, \ \sigma_{34} = \alpha_2,$$

$$V_{123} = V_1 \cdot V_2 \cdot V_3 = A_3^{-1}, \lambda_{123} = e^{2\pi i \alpha_1}, \mu_{123} = -e^{-\pi i \alpha_2},$$

$$\rho_{123} = 1 + \alpha_1, \sigma_{123} = (1 + \alpha_2)/2,$$

$$V_{234} = V_2 \cdot V_3 \cdot V_4 = A_1 A_4^{-1}, \lambda_{234} = -e^{-\pi i \alpha_1}, \mu_{234} = e^{2\pi i \alpha_2},$$

$$\rho_{234} = (1 + \alpha_1)/2, \sigma_{234} = 1 + \alpha_2.$$

The branches of the logarithms of monodromy matrix products are chosen from the conditions

$$\rho_{12} + \sigma_{12} = \rho_1 + \sigma_1 + \rho_2 + \sigma_2, \rho_{123} + \sigma_{123} = \rho_1 + \sigma_1 + \rho_2 + \sigma_2 + \rho_3 + \sigma_3$$

etc.

The behavior of the solution of problem (26) at infinity determine the numbers

$\rho = \rho_5 - 1 = -\alpha_1, \sigma = \sigma_5 - 2 = -\alpha_2 - 1$, if $\alpha_1 > \alpha_2$ and
$\rho = \rho_5 - 2 = -\alpha_1 - 1, \sigma = \sigma_5 - 1 = -\alpha_2$, if $\alpha_1 < \alpha_2$.

The numbers ρ_k,σ_k($k = 1, 2, 3, 4$),ρ,σ satisfy the Fuchs relation:

$$\sum_{k=1}^{4} (\rho_k + \sigma_k) + \rho + \sigma = 1.$$

The total index κ and partial indices æ$_1$, æ$_2$ of the problem (9) are respectively equal æ $= -\sum_{k=1}^{5} (\rho_k + \sigma_k) = -4$, æ$_1$ = æ$_2$ $= -2$, those problem (24) has a unique solution.

The canonical matrix $\boldsymbol{X}(x)$ of homogeneous boundary value problem

$$X^+(x) = A_k X^-(x), x \in l_k, \quad k = 1, 2, 3, 4,$$

is a solution of a system of differential equations of Fuchs class

$$\frac{dX}{dz} = X \sum_{k=1}^{4} \frac{U_k}{z - a_k}, \tag{27}$$

where $U_k \tilde{} \frac{1}{2\pi i} \ln V_k$, $k = 1, \ldots, 4$. Denote by

$$S = \begin{pmatrix} -\min(\rho, \sigma) & 0 \\ 0 & 1 - \max(\rho, \sigma) \end{pmatrix} = \begin{pmatrix} 1 + \alpha_1 & 0 \\ 0 & 1 + \alpha_2 \end{pmatrix}$$

and imagine the matrix S as the three sums of matrices:

$$S = S_1 + S_{234} = S_{123} + S_4 = S_{12} + S_{34}, \tag{28}$$

where

$$S_k \sim \frac{1}{2\pi i} \ln V_k, S_{12} \sim \frac{1}{2\pi i} \ln V_{12}, S_{34} \sim \frac{1}{2\pi i} \ln V_{34},$$

$$S_{234} \sim \frac{1}{2\pi i} \ln V_{234}, S_{123} \sim \frac{1}{2\pi i} \ln V_{123}.$$